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Asymptotic Theory of Beam Break-Up in Linear Accelerators*

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The phenomenon of radial beam break-up observed in terms of long, multi-section electron linear accelerators has been analyzed using a multi-cavity model. In this model the source of regeneration due to backward wave amplification is ignored. The dominant phenomenon involves build-up of radial modes in each cavity as coupled by the electron beam. The resulting differential equations are integrated by the method of steepest descent and by a numerical iteration method. Scaling laws in terms of the pulse length, beam intensity, energy gradient, and length of the structure are derived.

I. GENERAL DESCRIPTION OF OBSERVED PHENOMENA

THE observed beam current of the SLAC 960-section linac appears to obey the results of independent particle dynamics at low intensities. However, as was first observed on April 24, 1966, the pulse length of the transmitted beam appears to shorten, provided the beam current exceeds a threshold value at a given distance along the accelerator; the greater the distance, the lower the threshold. This general behavior is illustrated in Fig. 1. Further tests clearly indicated that the phenomenon responsible is the sudden onset of a radial progressive instability conventionally called beam break-up (BBU).

Observation of radial instability in high current linear accelerators is not new,[1–6] and the phenomenon has been conclusively associated with the excitation of transverse deflecting modes. However, one should clearly recognize that we are dealing with two quite distinct mechanisms by which such modes can lead to an amplifying action. The first mechanism discussed in the above references results from the negative group velocity of the HEM mode of the conventional disk-loaded structure. This negative group velocity will feed transverse energy from the end of a given accelerating section to the front, thus leading to the regenerative action involved in the "backward-wave oscillator." This phenomenon of regeneration within a given section characteristically occurs at currents of several hundred milliamperes at pulse lengths of several microseconds. The second mechanism which is dominant in a multisection relatively low current accelerator (such as SLAC or the Kharkov 2-GeV accelerator[7]) involves amplification from section to section, coupled only by the electron beam without backward propagation of electromagnetic energy.

In this paper we give the theory of the second mechanism only, which is the dominant cause of the BBU phenomena occurring at SLAC. As is seen, this mechanism is very general, being quite independent of the detailed structure of the accelerating sections.

II. THE MULTICAVITY MODEL

A. The Model

We represent each section of the accelerator by a single cavity; the cavity geometry constitutes a free parameter which can be chosen to fit the experimental behavior.

We assume:

(a) Only one resonant mode at a frequency ω_0 and loss factor Q is of significance.

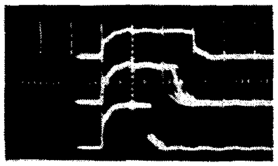
(b) The cavity has axial symmetry and the axial electric field vanishes along the axis of symmetry.

(c) The rate of build-up of oscillation giving rise to the radial modulation of the beam is small compared to ω_0.

Consider a particle of charge e to cross at a time t the nth of N cavities at a distance x from the z axis, taken to be an axis of symmetry. Let L be the distance between cavities, and let the particle velocity be $v\approx c=1$ (see Fig. 2).

B. Equation of Motion

Let the electric field **E** in the nth cavity be derivable from a vector potential **A**, and let each cavity be excited



0.5 μsec/ DIVISION ⟶

FIG. 1. Current pulse shapes observed at the end of the accelerator shown for 3 peak current amplitudes. Note the pulse shortening effect of beam break-up.

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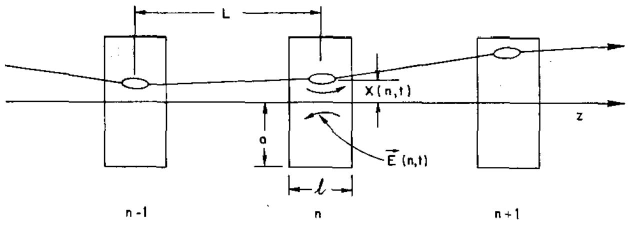

FIG. 2. The single cavity model. The figure shows the notation used describing the geometry of the nth cavity relative to the preceding and following unit. Each cavity represents resonant excitation of a particular transverse mode of each of the sections of the accelerator.

near a single resonant frequency ω_0. We obtain from the deflection theorem[8] for the change in transverse momentum p_x in the nth cavity

$$\Delta p_x = e\int \frac{\partial A_z}{\partial x}dz. \tag{1}$$

This transverse momentum Δp_x results in a difference in displacement of $(\Delta p_x/m_0\gamma)L$ between the $(n+1)$th and nth cavity where $m_0\gamma$ is the particle energy. We can thus write a difference equation which can be approximated as a transverse differential equation of motion as follows:

$$\frac{\partial}{\partial n}\left(\gamma\frac{\partial x}{dn}\right)=\frac{eL}{m_0}\int\frac{\partial A_z}{\partial x}dz. \tag{2}$$

C. Excitation of Cavities

Equation (2) gives the radial equation of motion as governed by the transverse gradient of the longitudinal component of the vector potential and hence the electric field. No special assumptions as to mode structure are assumed. If the particle passes at a distance x (assumed constant in each cavity) from the symmetry axis, then, in general, work is done against the longitudinal field.

Each cavity excited at a frequency ω near ω_0 loses energy U to the current j at the rate $j\int \mathbf{E}\cdot d\mathbf{z}$ and loses energy to wall losses at the rate $\omega U/Q$. The rate of build-up is therefore given by (averaged over many cycles as designated by the $\overline{}$ symbol)

$$\frac{\partial U}{\partial t}=-\overline{j\int \mathbf{E}\cdot d\mathbf{z}}-\frac{\omega U}{Q}. \tag{3}$$

If the field $\mathbf{E}=0$ on the axis and varies linearly from the axis, $\int\mathbf{E}\cdot d\mathbf{z}$ can be approximated by

$$\overline{\int \mathbf{E}\cdot d\mathbf{z}}\sim x\overline{\int\frac{\partial E_z}{\partial x}dz}, \tag{4}$$

where x is the transverse coordinate at which the beam carrying the current j passes from the axis. The cavity excitation is thus proportional to x; on the other hand the cavity, once excited, will affect the motion in x according to Eq. (2). As a result the beam will receive a transverse structure so that x will be modulated at a frequency ω near ω_0.

[8] W. K. H. Panofsky and W. A. Wentzel, Rev. Sci. Instr. **27**, 967 (1956).

Note that the field integrals appearing in Eq. (2) and in Eq. (4) are simply related if we can assume that oscillations take place near $\omega=\omega_0$ and that the rates of build-up or damping are slow relative to ω_0. We adopt the convention that all field amplitudes vary as $e^{+i\omega t}$ and that we consider the transverse displacement to vary as $e^{+i\omega t}$ also. The quantities x, E, A thus become complex amplitudes carrying *both* the phase and the (slowly varying) amplitude information.

Using this convention we can write the field integral in Eq. (4), using $E_z=-\partial A_z/\partial t$,

$$I(t)=\int\frac{\partial E_z}{\partial x}dz\cong -i\omega\int\frac{\partial A_z}{\partial x}dz, \tag{5}$$

giving the equations of motion and the energy build-up equations

$$(\partial/dn)[\gamma(\partial x/dn)]=(ieL/m_0\omega)I, \tag{6}$$

$$(\partial U/\partial t)+(\omega U/Q)=-\overline{xI}j. \tag{7}$$

In general U and I are related quadratically through a (generally complex) impedance. In general we can write

$$U=\tfrac{1}{2}Re\{KII^*\} \tag{8}$$

and

$$\overline{xI}=\tfrac{1}{2}Re\{xI^*\}, \tag{9}$$

where * denotes the complex conjugate and Re the real part. Hence Eq. (7) becomes,

$$(\partial I/\partial t)+(\omega/2Q)I=(-j/2K)x. \tag{10}$$

Combining Eqs. (6) and (10) we obtain:

$$\left[\left(\frac{\partial}{\partial t}+\frac{\omega}{2Q}\right)\frac{\partial}{\partial n}\left(\gamma\frac{\partial}{\partial n}\right)+\frac{ijeL}{2Km_0\omega}\right]x=0 \tag{11}$$

as the basic differential equation governing the build-up of the instability.

The impedance constant K can be related qualitatively to the dimensions of the cavity. Let l be the length of the cavity which can be interpreted as an effective "interaction length" in the actual case. We have dimensionally

$$U=\frac{1}{8\pi}\int|E|^2dv\sim(|I|/l)^2a^2\cdot(\pi a^2 l), \tag{12}$$

where a is a radial dimension of the cavity. More quantitatively, for a simple cylindrical cavity of radius $a=3.83/\kappa$

$$E_z=(2/\kappa)J_1(\kappa\rho)\cos\phi(\partial E_z/\partial x), \tag{13}$$

where the symbols have their usual meaning. The integral in Eq. (12) then gives

$$U=a^4|I|^2/362l. \tag{14}$$

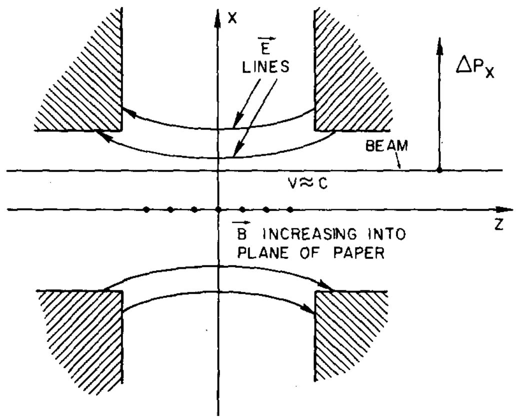

FIG. 3. Field configuration in a typical transverse mode.

The constant K in Eqs. (8)–(11) is thus

$$K=(a^4/l)/181. \tag{15}$$

Let us measure beam intensity in terms of the quantity J = number of particles/sec (or number of particles/unit length, since we take $v\approx c=1$). Hence we can write Eq. (11) as

$$\left[\left(\frac{\partial}{\partial t}+\beta\right)\frac{\partial}{\partial n}\left(\gamma\frac{\partial}{\partial n}\right)+(iCJ)\right]x=0 \tag{16}$$

where

$$\beta=\omega/2Q$$

and

$$C=14.4(e^2/m_0)(l\lambda L/a^4)=14.4(r_0 l\lambda L/a^4) \tag{17}$$

is a dimensionless constant expressed in terms of the classical electron radius $r_0=e^2/m_0=2.8\times10^{-13}$ cm.

D. Physical Discussion

The build-up of oscillation is governed by the integrals of Eq. (16); blow-up will be dominated by that frequency ω which will maximize the build-up rate.

Let us understand some of the qualitative features of these equations.

The field in the deflecting mode has the qualitative configuration shown in Fig. 3. The above analysis shows that the details of the field are of minimum significance, since the same integral I over the fields governs both the transverse momentum imparted to the particle as well as the coupling of the particle in "driving" the field build-up. Equation (6) shows that the transverse momentum Δp_x is in phase quadrature (leading) with the field integral I.

According to Eq. (10) a linear combination of the field integral I and its rate of build-up is 180° out of phase with the driving displacement x. On the other hand x and Δp_x must have a common in-phase component if the oscillations are to grow. Therefore for maximum build-up the phase of x will be somewhere between the phase of Δp_x and $-I$, as shown in Fig. 4.

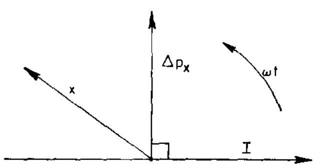

FIG. 4. Phase relationships between transverse displacement x, transverse momentum gain Δp_x, and the field integral **I**.

E. Solution by Laplace Transform Using the Method of Steepest Descent

In this section we study the solution of the Eq. (16) where γ is a given function of n.

Let us try the Laplace transform solution, using an appropriate contour c,

$$x(n,t)=e^{-\beta t}\int_c f(n,\mu)e^{\mu t}d\mu \tag{18}$$

in Eq. (16). The function $f(n,\mu)$ satisfies

$$\mu(\gamma f')'+iCJf=0 \tag{19}$$

where $'$ denotes differentiation with respect to n. We can integrate this equation by assuming adiabatic variation of γ with n (WKB approximation). The result is

$$f(n,\mu)\sim\gamma^{-\frac{1}{4}}\exp\{\pm i(iCJ)^{\frac{1}{2}}\mu^{-\frac{1}{2}}g\} \tag{20}$$

where

$$g(n)=\int_{n_0}^{n}\gamma^{-\frac{1}{2}}dn', \tag{21}$$

since the general WKB solution of the equation

$$(Af')'+Bf=0$$

is

$$f\sim(1/\sqrt[4]{AB})\exp\left[\pm i\int^n\sqrt{B/A}\,dn'\right]. \tag{22}$$

The i term in the exponent of Eq. (20) is carried from Eq. (16) and governs the phase of the space harmonic in x at the frequency ω relative to the phase of the electric field in each cavity, as discussed previously. Hence the general solution is

$$x(n,t)=e^{-\beta t}\int_c w(\mu)d\mu\exp\{\mu t\pm i[iCJ/\mu]^{\frac{1}{2}}g\}, \tag{23}$$

where the weighting function $w(\mu)$ depends on the starting conditions. We chose the root giving a positive real part in the exponent.

Evaluation of the function $w(\mu)$ in terms of specific starting condition, such as a unit disturbance in x occurring at $n=n_0$ at a fixed time, is quite straightforward, but evaluation of the inverse Laplace integral, Eq. (23), in general is not possible in closed analytical form.

Among such starting sources are shot noise in beam, shock excitation through misalignments, thermal noise in early sections, noise or spurious signals from klystron power sources, or electrical discharges in high microwave fields. Present experimental evidence is not conclusive as to which of these initial driving terms are important. However, the question of largest practical interest is the dependence of x on the various physical parameters (current, length of current pulse, number of cavities) once a "blow-up" of x, sufficiently large to lead to beam loss, has occurred. Such loss requires a large (10^7 to 10^9) amplification. For this purpose an asymptotic solution is adequate which can be generated by the method of steepest descent.

The "saddle point" of the exponent in Eq. (23) is at one of the roots $\mu=\epsilon$ of the equation

$$(d/d\mu)\{\pm(CJ)^{\frac{1}{2}}ge^{(\frac{3}{4})\pi i}\mu^{-\frac{1}{2}}+\mu t\}=0, \tag{24}$$

which occur at

$$\mu=\epsilon=(CJ)^{\frac{1}{3}}(g/2t)^{\frac{2}{3}}e^{(\frac{1}{2}+\frac{2}{3}n)\pi i}, \tag{25}$$

leading to a value $\theta(\epsilon)$ of the exponent of Eq. (23) of

$$\theta(\epsilon)=2^{\frac{1}{3}}t^{\frac{1}{3}}(CJ)^{\frac{1}{3}}g^{\frac{2}{3}}[\tfrac{3}{2}e^{(\frac{1}{2}+\frac{2}{3}n)\pi i}], \tag{26}$$

where n is any integer.

We chose that value of n for which ϵ has the greatest real part, i.e., for which blow-up occurs at the maximum rate. This gives $n=2$ or

$$\theta(\epsilon)=3(2^{\frac{1}{3}}/4)(\sqrt{3}-i)(CJt)^{\frac{1}{3}}g^{\frac{2}{3}}=(1.64-0.94i)(CJt)^{\frac{1}{3}}g^{\frac{2}{3}}. \tag{27}$$

The appropriate contour passes through the "saddle-point" $\mu=\epsilon$ along a direction of "steepest descent," i.e., along a direction to make $\theta(\mu)$ a greatest maximum. If we expand $\theta(\mu)$ about $\mu=\epsilon$, we can put

$$\theta(\mu)=\theta(\epsilon)+\tfrac{1}{2}(\mu-\epsilon)^2\theta''(\epsilon). \tag{28}$$

Differentiation of Eq. (26) gives

$$\theta''(\epsilon)=(3/2^{1/3})(CJ)^{-\frac{1}{3}}t^{5/3}g^{-\frac{2}{3}}e^{i\pi/6}. \tag{29}$$

Let the argument of the contour c of steepest descent when passing through $\mu=\epsilon$ be ψ, i.e., let $\mu-\epsilon=\rho e^{i\psi}$ where ρ and ψ are real. The argument of

$$(\mu-\epsilon)^2\theta''(\epsilon)=\rho^2|\theta''(\epsilon)|e^{i(2\psi+\pi/6)} \tag{30}$$

is π along the path of steepest descent, or $\psi=5\pi/12$. The appropriate contour is shown in Fig. 5. The function of (n,μ) thus falls off steepest in both directions along c away from the real axis. This leads to a useful approximation if the exponent is large, i.e., if "blow-up" has largely progressed.

The asymptotic solution is then given by evaluation of the integral [Eq. (23)], using Eqs. (26), (28), and (29); and obtain

$$x(n,t)=x_0(n,t)\exp[3(2^{\frac{1}{3}}/4)(\sqrt{3}-i)(CJt)^{\frac{1}{3}}g^{\frac{2}{3}}-\beta t], \tag{31}$$

where $x_0(n,t)$ is a relatively slowly varying function given by

$$x_0(n,t)\approx J^{1/6}t^{-5/6}g^{\frac{1}{3}}\gamma^{-\frac{1}{4}}. \tag{32}$$

The growth is thus controlled by the exponent

$$1.64C^{\frac{1}{3}}(tJg^2)^{\frac{1}{3}} \tag{33}$$

in the highly transient break-up observed at SLAC, where

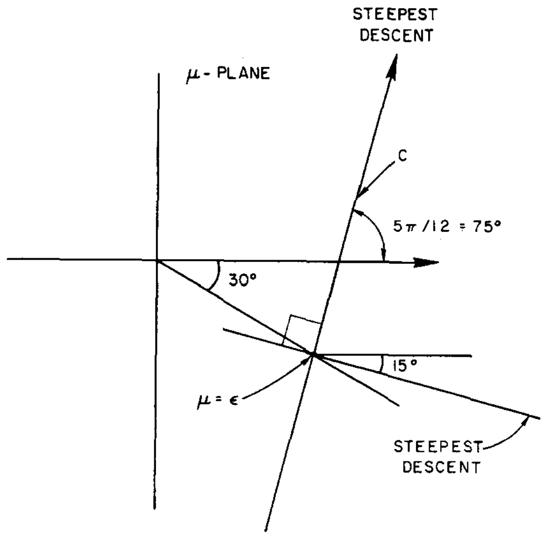

FIG. 5. Location of the saddle point in the complex plane and the path of steepest descent.

the term βt is small compared to the term given by Eq. (33).

For a constant "blow-up factor" we thus obtain the basic scaling law:

$$(Jt)g^2=\text{const.} \tag{34}$$

Hence the total charge (Jt) per pulse which can be accelerated to a specified point along the accelerator under a given acceleration program $\gamma(n)$ within a limiting blow-up factor is constant, i.e., independent of pulse length.

Let us evaluate the integral g if we accelerate from n_0 to n_1 at a uniform energy gain γ' and coast from there to section n_2. We obtain

$$g=\int_{n_0}^{n_2}\gamma^{-\frac{1}{2}}dn'=(2/\gamma'^{\frac{1}{2}})(n_1^{\frac{1}{2}}-n_0^{\frac{1}{2}}) +[1/(n_1\gamma')^{\frac{1}{2}}](n_2-n_1). \tag{35}$$

For constant acceleration (no coasting) and $n_1\gg n_0$ we obtain the simple scaling law,

$$Jtn_1/\gamma'=\text{const.}, \tag{36}$$

while for "pure" coasting from n_1 to n_2 at an energy γ_c we obtain

$$Jt(n_2-n_1)^2/\gamma_c=\text{const.} \tag{37}$$

Numerical comparison of these relations with experience is good and has been discussed elsewhere.[9] Suffice it to say that reasonable agreement is obtained with an exponent of about 20 leading to beam loss through BBU.

The steepest descent calculation leads to valid results only if the exponent is large. The error can be estimated by estimating the variation of $x_0(n,t)$ as given by Eq. (32) over the range of interest. For numerical situations of

[9] O. H. Altenmueller *et al.*, "Beam Break-Up Experiments at SLAC," Stanford Linear Accelerator Center, Stanford Univ., Stanford, Calif., Proc. 1966 Linear Accelerator Conf. LA-3609 (3–7 Oct.. 1966).

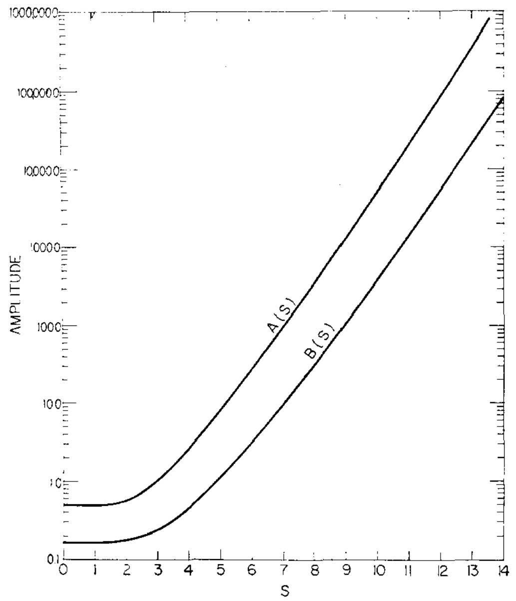

FIG. 6. The amplitudes of the functions A(S) and B(S) as defined in Eq. (49).

interest this might add a correction of no more than 20% to the exponent.

F. Solution by Iteration

The steepest descent solution is purely asymptotic and thus cannot be linked to the starting condition. Let us now examine the solution of the basic differential Eq. (16) under the boundary conditions corresponding to a δ-function impulse at $t=T$, i.e.,

$$x(n=n_0,\, t)=\delta(t-T),$$
$$(\partial x/\partial n)(n=n_0,\, t)=0. \tag{38}$$

As mentioned above, the exponential factor $\exp(-\beta t)$ is always factorable; we can thus write a first integral by putting

$$y(n,t)=x(n,t)\,\exp(\beta t), \tag{39}$$

where $y(n,t)$ satisfies

$$\frac{\partial}{\partial n}\left(\gamma\frac{\partial y}{\partial n}(n,t)\right)+iCJ\int_{-\infty}^{t} y(n,t')dt'=0, \tag{40}$$

and where the boundary conditions are

$$y(n=n_0,\, t)=\delta(t-T)e^{\beta T}$$
$$(\partial y/\partial n)(n=n_0,\, t)=0. \tag{41}$$

We again make the WKB approximation. Let

$$y(n,t)=\frac{1}{[\gamma(n)]^{\frac{1}{4}}}F[g(n),t]. \tag{42}$$

If we neglect terms of the order of $(d\gamma/dn)^2$ and $(d^2\gamma/dn^2)$ (which is exact in the absence of acceleration), we find that $F(z,t)$ satisfies

$$(\partial^2/\partial z^2)F(z,t)=-iCJ\int_{-\infty}^{t} F(z,t')dt'$$
$$F(0,t)=\gamma_0^{\frac{1}{4}}e^{\beta T}\delta(t-T) \tag{43}$$
$$(\partial F/\partial z)(z=0,\, t)=(\gamma_0'/4\gamma_0^{\frac{1}{4}})\delta(t-T)e^{\beta T}$$
$$\gamma_0=\gamma(n_0)\,;\ \gamma_0'=(\partial\gamma/\partial n)(n=n_0).$$

Incorporating these boundary conditions we obtain an integral equation for $F(z,t)$,

$$F(z,t)=-iCJ\int_{-\infty}^{t}dt'\int_0^z dz'\int_0^{z'}dz''F(z'',t')$$
$$+(\gamma_0'/4\gamma_0^{\frac{1}{4}})ze^{\beta T}\delta(t-T)+\gamma_0^{\frac{1}{4}}\delta(t-T)e^{\beta T}. \tag{44}$$

The solution Eq. (7) may be found by iteration,

$$F(z,t)=e^{\beta t}\delta(t-T)\left(\gamma_0^{\frac{1}{4}}+\frac{\gamma_0'z}{4\gamma_0^{\frac{1}{4}}}\right)$$
$$+\theta(t-T)e^{\beta T}\sum_{j=1}^{\infty}(-iCJ)^j\frac{(t-T)^{j-1}}{(j-1)!}$$
$$\times\left[\frac{\gamma_0^{\frac{1}{4}}z^{2j}}{(2j)!}+\frac{\gamma_0'}{4\gamma_0^{\frac{1}{4}}}\frac{z^{2j+1}}{(2j+1)!}\right], \tag{45}$$

where $\theta(t-T)$ is a unit step function.

Using the definition for

$$g(n)=\int_{n_0}^{n}\gamma^{-\frac{1}{2}}dn'$$

given by Eq. (21), we may trace back to obtain a solution for $x(n,t)$,

$$x(n,t)=\frac{e^{-\beta(t-T)}}{[\gamma(n)]^{\frac{1}{4}}}\left\{\left(\gamma_0^{\frac{1}{4}}+\frac{\gamma_0'}{4\gamma_0^{\frac{1}{4}}}g(n)\right)\delta(t-T)\right.$$
$$+\theta(t-T)\sum_{j=1}^{\infty}(-iCJ)^j\frac{(t-T)^{j-1}}{(j-1)!}$$
$$\left.\times\left[\frac{\gamma_0^{\frac{1}{4}}[g(n)]^{2j}}{(2j)!}+\frac{\gamma_0'}{4\gamma_0^{\frac{1}{4}}}\frac{[g(n)]^{2j+1}}{(2j+1)!}\right]\right\}. \tag{46}$$

For the coasting (nonaccelerating) case this simplifies to

$$x(n,t)=e^{-\beta(t-T)}\left\{\delta(t-T)+\theta(t-T)\frac{CJ}{\gamma_c}(n-n_0)^2\right.$$
$$\left.\times\sum_{j=1}^{\infty}(-i)^j\left(\left[(t-T)(n-n_0)^2\frac{CJ}{\gamma_0}\right]^{j-1}\Big/\left[(2j)!(j-1)!\right]\right)\right\}. \tag{47}$$

Due to the appearance of $(2j)!(j-1)!$ in the denominator, this series converges very rapidly. For $z^2(t-T)\sim 3000$, fifteen terms would be sufficient.

In Figs. 6 and 7, the numerical evaluation of the sums is presented. Let $g(n)$, γ_0, and γ_0' be as above. Note that

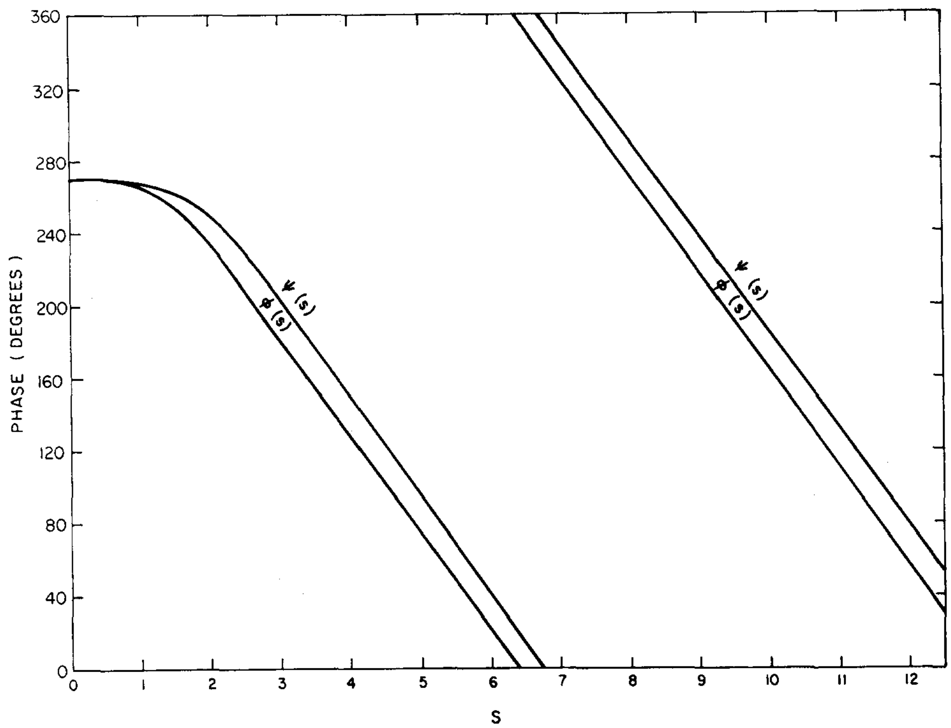

FIG. 7. The phases ψ(s) and ϕ(s) as defined in Eq. (49).

the expansion parameter

$$s^3=CJg^2(t-T) \tag{48}$$

is the same as that appearing in the asymptotic calculation leading to Eq. (31). Re-expressing Eq. (46) in terms of functions of s, we obtain

$$x(n,t)=\frac{e^{-\beta(t-T)}}{[\gamma(n)]^{1/4}}\left\{\left(\gamma_0^{1/4}+\frac{\gamma_0'}{4\gamma_0^{3/4}}g(n)\right)\delta(t-T)+\theta(t-T)CJg^2 \times\left[\gamma_0^{1/4}A(s)e^{i\phi(s)}+\frac{\gamma_0'g(n)}{4\gamma_0^{3/4}}B(s)e^{i\psi(s)}\right]\right\}. \tag{49}$$

The amplitude functions $A(s)$ and $B(s)$ are shown in Fig. 6. Asymptotically the functions behave as

$$\exp(3^{3/2}\times 2^{1/3}/4)s=\exp(1.64s),$$

in agreement with the steepest descent solution Eqs. (27) and (31). Figure 7 gives the phase functions $\phi(s)$ and $\psi(s)$. The first term in Eq. (49) is generally negligible since it represents the original impulse without build-up.

These figures (Figs. 6 and 7) can be used to construct by superposition any build-up pattern resulting from an initial disturbance $x(t)$ at $n=n_0$.

G. Steady State Solution

If the pulse length is sufficiently long, equilibrium with the wall losses will be reached. The basic differential equation (16) then reduces to

$$\left[\beta\frac{\partial}{\partial n}\left(\gamma\frac{\partial}{\partial n}\right)+iCJ\right]x=0. \tag{50}$$

This has the WKB solution [see Eq. (21)]

$$x(n)\sim(\beta/\gamma CJ)^{1/4}\exp\left\{\pm i\int^n(iCJ/\beta\gamma)^{1/2}dn'\right\}. \tag{51}$$

Ignoring a phase factor and multiplicative constants, this gives a positive exponential solution, valid at a time $t\gg Q/\omega$,

$$x(n)\sim[\gamma(n)]^{-1/4}\exp[(QCJ/\omega)^{1/2}g(n)]. \tag{52}$$

This case is not of relevance to the SLAC accelerator since $Q/\omega\sim 1$ μsec. However, for a potential low temperature accelerator permitting cw operation, the steady state solution is of interest.

III. THE EFFECT OF TRANSVERSE FOCUSING

In the previous sections we analyzed the general behavior of the BBU phenomenon in the absence of transverse forces other than those associated with the transverse modes associated with the BBU itself. The actual accelerator contains a series of strong focusing lenses to confine the beam; these will affect the BBU "gain" of each section and their strength and distribution can be used to increase the BBU current threshold by a significant amount.

The theory of the preceding section is completely linear; it is therefore easy to introduce the effect of linear focusing devices such as quadrupole or magnetic solenoids; on the other hand it is difficult to introduce either the effect of lenses of higher than quadrupole order, or the effect of bunching into the theory; we note however that to the extent that the bunch structure is incoherent with the frequency of the BBU, longitudinal bunching will not affect the phenomenon.

On the basis of these remarks we can introduce the effect of linear transverse focusing by introducing a term

$$\gamma k^2 x \tag{53}$$

into the basic differential Eq. (19), giving

$$\mu[(\gamma f')' + \gamma k^2 f] + iCJf = 0. \tag{54}$$

Here $2\pi/k(n)$ is the "betatron wavelength" produced by the external focusing system produced by quadrupoles.

The Laplace transform solution [Eq. (22)] then becomes

$$f(n,\mu) = \gamma^{-\frac{1}{2}} \exp\left[\pm i \int_{n_0}^{n} (iCJ/\mu\gamma + k^2)^{\frac{1}{2}} dn' \right]. \tag{55}$$

The evaluation of this integral by the method of steepest descent is not possible analytically. We can however obtain an approximate solution for "weak focusing" corresponding to

$$k^2 \ll CJ/\epsilon\gamma \tag{56}$$

where ϵ is given by Eq. (25). Carrying only linear terms in k^2, the exponent in Eq. (55) becomes

$$\theta(\mu) = \mu t \pm [i^{\frac{3}{2}} (CJ/\mu)^{\frac{1}{2}} g + i^{-\frac{1}{2}} (\mu/CJ)^{\frac{1}{2}} K], \tag{57}$$

where $g(n)$ is given by Eq. (21) as before, and K is the integral

$$K(n) = \frac{1}{2} \int_{n_0}^{n} k^2 \gamma^{\frac{1}{2}} dn'. \tag{58}$$

The saddle point occurs to order linear in K at the point $\mu = \epsilon$ given by

$$\epsilon = (CJ)^{\frac{1}{3}} (g/2t)^{\frac{2}{3}} e^{-\pi i/6} [1 - (2^{\frac{1}{3}}/3) K g^{-\frac{1}{3}} (CJt)^{-\frac{2}{3}} e^{-(\frac{2}{3})\pi i}]. \tag{59}$$

This leads to a value of the exponent $\theta(\epsilon)$ given by

$$\theta(\epsilon) = \tfrac{3}{2} \times 2^{\frac{1}{3}} (g^2 CJt)^{\frac{1}{3}} e^{-(\pi i/6)} - 2^{-\frac{1}{3}} g^{\frac{1}{3}} K (CJt)^{-\frac{1}{3}} e^{\pi i/6}. \tag{60}$$

The leading term agrees with Eq. (27) while the second term is a damping factor proportional to the focusing integral K. The real part of Eq. (60) can be written in the simple form

$$Re[\theta(\epsilon)] = F\left[1 - \frac{0.56}{F^2} \left(\int^{n} \gamma^{-\frac{1}{2}} dn' \right) \left(\int_{n_0}^{n} \gamma^{\frac{1}{2}} k^2 dn' \right) \right], \tag{61}$$

where $F = 1.64(g^2 CJt)^{\frac{1}{3}}$ is the exponent in the absence of external focusing. The current which will lead to a given value of BBU amplification will thus be increased by a factor f given by

$$f \sim 1 + (3 \times 0.56/F^2) \left(\int_{n_0}^{n} \gamma^{-\frac{1}{2}} dn' \right) \left(\int_{n_0}^{n} \gamma^{\frac{1}{2}} k^2 dn' \right). \tag{62}$$

This formula gives reasonable agreement with experiment for a value of $F \approx 20$.

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